PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the **focus**.

(b) The fixed straight line is called the **directrix**.

(c) The constant ratio is called the **eccentricity** denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the **axis**.

(e) A point of intersection of a conic with its axis is called a **vertex**.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will coincident.

$e < 1$ the lines will be imaginary.

Case (ii) When the focus does not lie on the directrix :

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1$; $D \neq 0$ $h^2 = ab$	$0 < e < 1$; $D \neq 0$ $h^2 < ab$	$D \neq 0$; $e > 1$; $h^2 > ab$	$e > 1$; $D \neq 0$ $h^2 > ab$; $a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is $(0, 0)$ (ii) Focus is $(a, 0)$ (iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the **focal distance of the point.**

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a **focal chord**.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a **double ordinate**.

(d) Latus rectum :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **latus rectum**. For $y^2 = 4ax$.

- Length of the latus rectum = **4a.**
- Length of the semi latus rectum = **2a**.
- Ends of the latus rectum are **L(a, 2a) & L'(a, − 2a)**

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

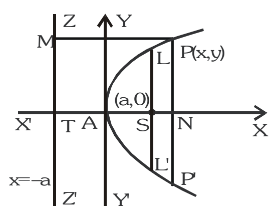
(iii) Two parabolas are said to be equal if they have the same latus rectum.

5. PARAMETRIC REPRESENTATION :

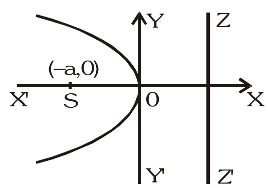
The simplest & the best form of representing the co-ordinates of a point on the parabola is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. TYPE OF PARABOLA :

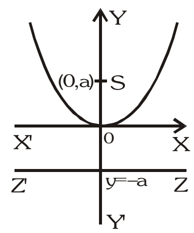
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



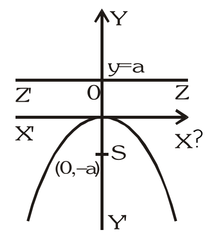
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=−a	4a	(a, ±2a)	$(at^2,2at)$	x + a
$y^2 = -4ax$	(0,0)	(−a,0)	y=0	x=a	4a	(−a, ±2a)	$(-at^2,2at)$	x−a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=−a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,−a)	x=0	y=a	4a	(±2a, −a)	$(2at, -at^2)$	y−a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a− h =0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x−h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b−q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y−q+b

Illustration 1 : Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Solution : The given equation can be rewritten as $\left(y-\frac{2}{3}\right)^2 = \frac{16}{9}\left(x+\frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$.

Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$

The axis is $y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$

The directrix is $X + A = 0 \Rightarrow x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow x = -\frac{613}{144}$

The focus is $X = A$ and $Y = 0 \Rightarrow x + \frac{61}{16} = \frac{4}{9}$ and $y - \frac{2}{3} = 0$

$\Rightarrow$ focus = $\left(-\frac{485}{144}, \frac{2}{3}\right)$

Length of the latus rectum = $4A = \frac{16}{9}$

The tangent at the vertex is X = 0 $\Rightarrow$ $x = -\frac{61}{16}$. **Ans.**

Illustration 2 : The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$

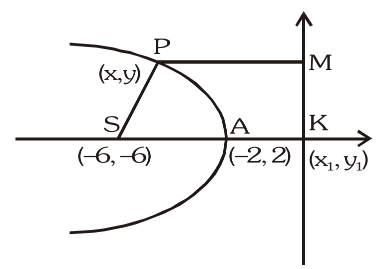
Solution : The length of latus rectum = 2 perp. from focus to the directrix

$$= 2 \times \left|\frac{2-4(3)+3}{\sqrt{(1)^2+(4)^2}}\right| = \frac{14}{\sqrt{17}}$$ **Ans. (D)**

Illustration 3 : Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

Solution : Let S(–6, –6) be the focus and A(–2, 2) is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that SA = AK. Draw KM perpendicular on SK. Then KM is the directrix of the parabola. Since A bisects SK, $\left(\frac{-6+x_1}{2}, \frac{-6+y_1}{2}\right) = (-2, 2)$

$\Rightarrow$ $-6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$



Hence the equation of the directrix KM is

$y - 10 = m(x - 2)$ .... (i)

Also gradient of SK = $\frac{10-(-6)}{2-(-6)} = \frac{16}{8} = 2$; $\Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x-2)$ (from (i))

$\Rightarrow$ $x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola. From SP = PM, the equation of the parabola is $\sqrt{\left\{(x+6)^2+(y+6)^2\right\}} = \frac{|x+2y-22|}{\sqrt{(1^2+2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$

or $(2x - y)^2 + 104x + 148y - 124 = 0$. **Ans.**

Illustration 4 : The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Solution : Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y - 4 = \frac{0}{5-3}(x-7) \Rightarrow y = 4$$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightwards and the other concave leftwards.

The vertex of the first parabola is (6.5, 4) and its equation is $(y - 4)^2 = 2(x - 6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y - 4)^2 = -2(x - 7.5)$. It meets the x-axis at (–0.5, 0). **Ans.**

Do yourself - 1 :

(i) Name the conic represented by the equation $\sqrt{ax}+\sqrt{by}=1$, where a, b $\in$ R, a, b, > 0.

(ii) Find the vertex, axis, focus, directrix, latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

(iii) Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

(iv) Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y = 4$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.

7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies **outside**, **on** or **inside** the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is **positive**, **zero** or **negative**.

Illustration 5 : Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Solution :

$\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore$ $y_1^2 - 4ax_1 < 0$

$\Rightarrow$ $\alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow$ $\alpha^2 - 4\alpha + 4 < 0$

$(\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$ **Ans.**

8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is

$$y(t_1 + t_2) = 2x + 2at_1t_2$$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Illustration 6 : Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Solution : The given parabola is $y^2 = 4x$ ...... (i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$

Slope of OP $= \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of OQ $= \frac{2}{t_2}$

Since OP $\perp$ OQ, $\frac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ ...... (ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow$ $y\left(t_1 - \frac{4}{t_1}\right) = 2(x - 4)$ [from (ii)]

$\Rightarrow$ $2(x - 4) - y\left(t_1 - \frac{4}{t_1}\right) = 0 \Rightarrow L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$

i.e. (4, 0) **Ans.**

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm \Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : Length of the focal chord making an angle α with the x - axis is $4a\,\mathrm{cosec}^2\,\alpha$.

Illustration 7 : If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

(A) $(3, \infty)$ (B) $(-\infty, 1/3)$ (C) $(1/3, 3)$ (D) none of these

Solution : Putting value of y from the line in the parabola -

$(3x + \lambda)^2 = 4x$

$\Rightarrow$ $9x^2 + (6\lambda - 4)x + \lambda^2 = 0$

$\because$ line cuts the parabola at two distinct points

$\therefore$ $D > 0$

$\Rightarrow$ $4(3\lambda - 2)^2 - 4.9\lambda^2 > 0$

$\Rightarrow$ $9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0$

$\Rightarrow$ $\lambda < 1/3$

Hence, $\lambda \in (-\infty, 1/3)$ **Ans.(B)**

Do yourself - 2 :

(i) Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

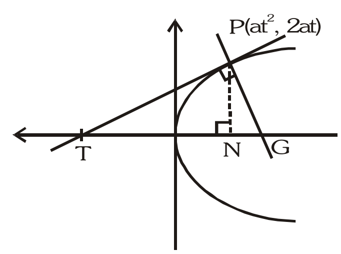
(ii) The focal distance of a point on the parabola $(x - 1)^2 = 16(y - 4)$ is 8. Find the co-ordinates.

(iii) Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with x-axis is of length $4a\,\mathrm{cosec}^2\alpha$.

(iv) Find the condition that the straight line $ax + by + c = 0$ touches the parabola $y^2 = 4kx$.

(v) Find the length of the chord of the parabola $y^2 = 8x$, whose equation is $x + y = 1$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P
(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$yy_1 = 2a(x + x_1)$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m},\ (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) **Parametric form :**

Equation of tangent to the given parabola at its point P(t), is **$ty = x + at^2$**

Note : Point of intersection of the tangents at the point **t_1 & t_2 is [at_1t_2, $a(t_1 + t_2)$].**

Illustration 8 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45 with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution : Let the slope of the tangent be m

$\therefore \quad \tan 45 = \left|\frac{3-m}{1+3m}\right| \Rightarrow \quad 1+3m = \pm(3-m)$

$\therefore \quad m = -2 \quad \text{or} \quad \frac{1}{2}$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

for m = –2, equation of tangent is $y = -2x - 1$ and point of contact is $\left(\frac{1}{2}, -2\right)$

for $m = \frac{1}{2}$, equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is (8, 8) **Ans.**

Illustration 9 : Find the equation of the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is

$y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\therefore \quad 10 = 4m + \frac{9}{4m} \Rightarrow \quad 16m^2 - 40\,m + 9 = 0$

$m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$ **Ans.**

Illustration 10 : Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that -

(i) $m_1^2 + m_2^2 = \lambda$ (constant) (ii) $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

(i) $m_1^2 + m_2^2 = \lambda$

$(m_1 + m_2)^2 - 2m_1m_2 = \lambda$

$\frac{k^2}{h^2} - 2.\frac{a}{h} = \lambda$

$\therefore$ locus of P(h, k) is $y^2 - 2ax = \lambda x^2$

(ii) $\theta_1 - \theta_2 = \theta_0$

$\tan(\theta_1 - \theta_2) = \tan\theta_0$

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \tan\theta_0$$

$(m_1 + m_2)^2 - 4m_1m_2 = \tan^2\theta_0(1 + m_1m_2)^2$

$$\frac{k^2}{h^2} - \frac{4a}{h} = \tan^2\theta_0\left(1 + \frac{a}{h}\right)^2$$

$k^2 - 4ah = (h + a)^2 \tan^2\theta_0$

$\therefore$ locus of P(h, k) is $y^2 - 4ax = (x + a)^2\tan^2\theta_0$ **Ans.**

Do yourself - 3 :

(i) Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

(ii) Find the equation of the tangents to the parabola $y^2 = 16x$, which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$. Find also the co-ordinates of their points of contact.

(iii) Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle θ is $(x + a)^2 \tan^2\theta = y^2 - 4ax$.

12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$

foot of the normal is $(am^2, -2am)$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2),\ -at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $am^3 + m(2a - h) + k = 0$.

This gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$; $m_1m_2m_3 = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

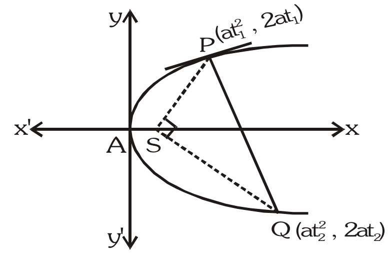
- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).

Illustration 11 : Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Solution : Let the normal at $P\left(at_1^2, 2at_1\right)$ meet the curve at $Q\left(at_2^2, 2at_2\right)$

$\therefore$ PQ is a normal chord.

and $t_2 = -t_1 - \dfrac{2}{t_1}$ ....(i)



By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a)

but focus S(a, 0)

$\therefore$ Slope of SP $= \dfrac{4a-0}{4a-a} = \dfrac{4a}{3a} = \dfrac{4}{3}$

and Slope of SQ $= \dfrac{-6a-0}{9a-a} = \dfrac{-6a}{8a} = -\dfrac{3}{4}$

$\therefore$ Slope of SP Slope of SQ $= \dfrac{4}{3} \times -\dfrac{3}{4} = -1$

$\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Illustration 12 : If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha \,.\, \tan\beta = 2$, then find the locus of this point.

Solution : Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

passes through (h, k)

$k = mh - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation, then $m_1 . m_2 . m_3 = -\dfrac{k}{a}$

but $m_1m_2 = 2$, $m_3 = -\dfrac{k}{2a}$

m_3 is root of (i) $\therefore$ $a\left(-\dfrac{k}{2a}\right)^3 - \dfrac{k}{2a}(2a-h) + k = 0 \Rightarrow k^2 = 4ah$

Thus locus is $y^2 = 4ax$. **Ans.**

Illustration 13 : Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the coordinates of the feet of the normals.

Solution : The given parabola is $y^2 - 16x - 8y = 0$ .... (i)

Let the co-ordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$. Now the equation of the tangent at $P(\alpha, \beta)$ to parabola (i) is

$y\beta - 8(x + \alpha) - 4(y + \beta) = 0$

or $(\beta - 4)y = 8x + 8a + 4\beta$ .... (ii)

Its slope $= \dfrac{8}{\beta - 4}$

Equation of the normal to parabola (i) at (α, β) is $y - \beta = \dfrac{4-\beta}{8}(x - \alpha)$

It passes through (14, 7)

$\Rightarrow \quad 7 - \beta = \dfrac{4-\beta}{8}(14 - \alpha) \Rightarrow \alpha = \dfrac{6\beta}{\beta - 4}$ .... (iii)

Also (α, β) lies on parabola (i) i.e. $\beta^2 - 16\alpha - 8\beta = 0$ .... (iv)

Putting the value of α from (iii) in (iv), we get $\beta^2 - \dfrac{96\beta}{\beta-4} - 8\beta = 0$

$\Rightarrow \quad \beta^2(\beta - 4) - 96\beta - 8\beta(\beta - 4) = 0 \Rightarrow \beta(\beta^2 - 4\beta - 96 - 8\beta + 32) = 0$

$\Rightarrow \quad \beta(\beta^2 - 12\beta - 64) = 0 \Rightarrow \beta(\beta - 16)(\beta + 4) = 0$

$\Rightarrow \quad \beta = 0, 16, -4$

from (iii), $\alpha = 0$ when $\beta = 0$; $\alpha = 8$, when $\beta = 16$; $\alpha = 3$ when $\beta = -4$

Hence the feet of the normals are (0, 0), (8, 16) and (3, –4) **Ans.**

Do yourself - 4 :

(i) If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

(ii) Find the number of distinct normal that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

(iii) If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

(iv) Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the co-ordinates of the feet of the normals.

13. AN IMPORTANT CONCEPT :

If a family of straight lines can be represented by an equation $\boldsymbol{\lambda^2 P + \lambda Q + R = 0}$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $\boldsymbol{Q^2 = 4PR}$.

Illustration 14 : If the equation $m^2(x + 1) + m(y - 2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Solution : $m^2(x + 1) + m(y - 2) + 1 = 0$

The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

$\therefore \quad (y - 2)^2 - 4(x + 1) = 0$

$y^2 - 4y + 4 - 4x - 4 = 0$

$y^2 = 4(x + y)$ **Ans.**

14. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $\mathbf{SS_1 = T^2}$ where :

$\mathbf{S \equiv y^2 - 4ax}$; $\mathbf{S_1 \equiv y_1^2 - 4ax_1}$; $\mathbf{T \equiv yy_1 - 2a(x + x_1)}$.

15. DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **director circle.** It's equation is $\mathbf{x + a = 0}$ which is parabola's own directrix.

Illustration 15 : The angle between the tangents drawn from a point (–a, 2a) to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

Solution : The given point (–a, 2a) lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle. **Ans. (B)**

Illustration 16 : The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

(A) $x + 5 = 0$ (B) $y + 5 = 0$ (C) $x + y + 5 = 0$ (D) $x - y + 5 = 0$

Solution : Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e. the line $x + 5 = 0$. **Ans.(A)**

Do yourself - 5 :

(i) If the equation $\lambda^2x + \lambda y - \lambda^2 + 2\lambda + 7 = 0$ represents a family of lines, where 'λ' is parameter, then find the equation of the curve to which these lines will always be tangents.

(ii) Find the angle between the tangents drawn from the origin to the parabola, $y^2 = 4a(x - a)$.

16. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Note : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$ i.e. $\frac{\left(S_1\right)^{3/2}}{2a}$, **also note that the chord of contact exists only if the point P is not inside.**

Illustration 17 : If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0$ .... (i)

But given line is

$x - y - 1 = 0$ .... (ii)

Comparing (i) and (ii)

$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \quad \Rightarrow \quad h = -1,\ k = 4$

$\therefore$ point $\equiv (-1, 4)$ **Ans.**

Illustration 18 : Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ .... (i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$y = \frac{2b}{k}x + \frac{2bh}{k}$ .... (ii)

From (i) & (ii)

$m = \frac{2b}{k},\ \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$

locus of P is $y^2 = \frac{4b^2}{a}x$. **Ans.**

---

17. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}\left(x - x_1\right)$.

This reduced to $T = S_1$, where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

Illustration 19 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Solution : Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore \quad qk - 2a(p + h) = k^2 - 4ah$

$\therefore \quad$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Illustration 20 : Find the locus of the middle point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Solution : The equation of the chord of the parabola whose middle point is (α, β) is

$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha$

$\Rightarrow \quad y\beta - 2ax = \beta^2 - 2a\alpha$

or $\quad \dfrac{y\beta - 2ax}{\beta^2 - 2a\alpha} = 1 \qquad$ .... (i)

Now, the equation of the pair of the lines OP and OQ joining the origin O i.e. the vertex to the points of intersection P and Q of the chord with the parabola $y^2 = 4ax$ is obtained by making the equation homogeneous by means of (i). Thus the equation of lines OP and OQ is

$$y^2 = \frac{4ax(y\beta - 2ax)}{\beta^2 - 2a\alpha}$$

$\Rightarrow \quad y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

If the lines OP and OQ are at right angles, then the coefficient of x^2 + the coefficient of $y^2 = 0$

Therefore, $\beta^2 - 2a\alpha + 8a^2 = 0 \Rightarrow \beta^2 = 2a(\alpha - 4a)$

Hence the locus of (α, β) is $y^2 = 2a(x - 4a)$

Do yourself - 6 :

(i) Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

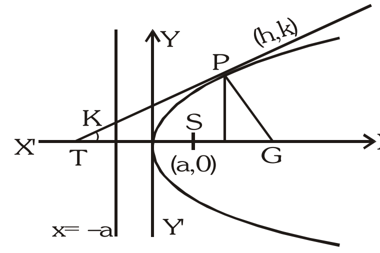
(ii) Find the co-ordinates of the middle point of the chord of the parabola $y^2 = 16x$, the equation of which is $2x - 3y + 8 = 0$

(iii) Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangent at the extremities of the chords are perpendicular.

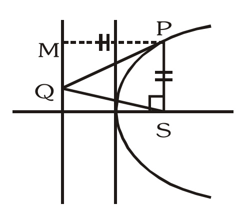
18. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is **$y = 2a/m$**, where **m = slope of parallel chords**.

19. IMPORTANT HIGHLIGHTS :



(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

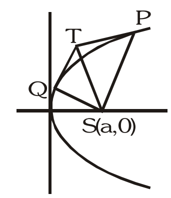


(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $2a = \frac{2bc}{b+c}$ i.e. $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

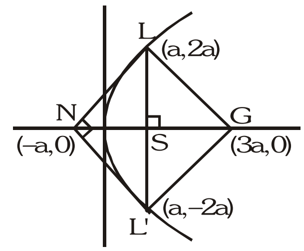


(f) If the tangents at P and Q meet in T, then :

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP \cdot SQ$ &

(iii) The triangles SPT and STQ are similar.



(g) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.

Note :

(i) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(ii) Figure LNL'G is square of side $2\sqrt{2}a$

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

Do yourself - 7 :

(i) The parabola $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the line $x = 4$, $y = 4$ and the co-ordinates axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then find $S_1 : S_2 : S_3$.

(ii) Let P be the point (1, 0) and Q a point on the parabola $y^2 = 8x$, then find the locus of the mid point of PQ.

Miscellaneous Illustrations :

Illustration 21 : The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is -

(A) $y = x + a$ (B) $x + y + a = 0$ (C) $x + y + 2a = 0$ (D) $y = x + 2a$

Solution : Any tangent to parabola is $y = mx + \frac{2a}{m}$

Solving with the circle $x^2 + (mx + \frac{2a}{m})^2 = 2a^2 \Rightarrow x^2(1 + m^2) + 4ax + \frac{4a^2}{m^2} - 2a^2 = 0$

$B^2 - 4AC = 0$ gives $m = \pm 1$

Tangent $y = \pm x \pm 2a$ **Ans. (C,D)**

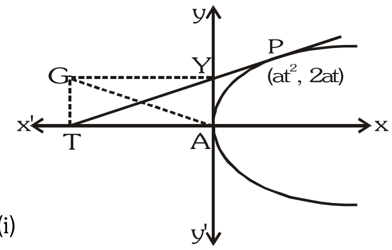
Illustration 22 : If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Solution : Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then tangent at $P(at^2, 2at)$ is $ty = x + at^2$

Since tangent meet the axis of parabola in T and tangent at the vertex in Y.

$\therefore$ Co-ordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.



Let co-ordinates of G be (x_1, y_1).

Since TAYG is rectangle.

$\therefore$ Mid-points of diagonals TY and GA is same

$\Rightarrow \dfrac{x_1+0}{2} = \dfrac{-at^2+0}{2} \Rightarrow x_1 = -at^2$ .... (i)

and $\dfrac{y_1+0}{2} = \dfrac{0+at}{2} \Rightarrow y_1 = at$ .... (ii)

Eliminating t from (i) and (ii) then we get $x_1 = -a\left(\dfrac{y_1}{a}\right)^2$

or $y_1^2 = -ax_1$ or $y_1^2 + ax_1 = 0$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$

Illustration 23 : If P(–3, 2) is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is -

(A) –1/2 (B) 2 (C) 1/2 (D) –2

Solution : The equation of the tangent at (–3, 2) to the parabola $y^2 + 4x + 4y = 0$ is

$2y + 2(x - 3) + 2(y + 2) = 0$

or $2x + 4y - 2 = 0 \Rightarrow x + 2y - 1 = 0$

Since the tangent at one end of the focal chord is parallel to the normal at the other end, the slope of the normal at the other end of the focal chord is $-\dfrac{1}{2}$. **Ans.(A)**

Illustration 24 : Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.

Solution : Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are $y = mx - 2am - am^3$ and $y = m(x - b) - 2cm - cm^3$ respectively then normals must be identical, compare the co-efficients

$$1 = \frac{2am + am^3}{mb + 2cm + cm^3}$$

$\Rightarrow m[(c - a)m^2 + (b + 2c - 2a)] = 0,\ m \neq 0$ ($\because$ other than axis)

and $m^2 = \dfrac{2a-2c-b}{c-a},\ m = \pm\sqrt{\dfrac{2(a-c)-b}{c-a}}$

or $m = \pm\sqrt{\left(-2 - \dfrac{b}{c-a}\right)}$

$\therefore -2 - \dfrac{b}{c-a} > 0$

or $-2 + \dfrac{b}{a-c} > 0 \Rightarrow \dfrac{b}{a-c} > 2$

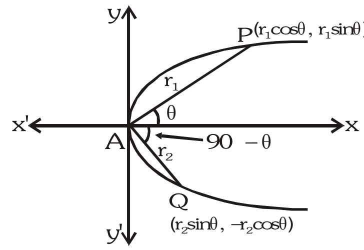
Illustration 25 : If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1r_2)^{4/3} = 16a^2\left(r_1^{2/3} + r_2^{2/3}\right)$.

Solution : Since chord are perpendicular, therefore if one makes an angle θ then the other will make an angle (90 – θ) with x-axis

Let $AP = r_1$ and $AQ = r_2$

If $\angle PAX = \theta$

then $\angle QAX = 90 - \theta$



$\therefore$ Co-ordinates of P and Q are $(r_1 \cos\theta,\ r_1 \sin\theta)$

and $(r_2 \sin\theta,\ -r_2 \cos\theta)$ respectively.

Since P and Q lies on $y^2 = 4ax$

$$\therefore \quad r_1^2 \sin^2\theta = 4ar_1 \cos\theta \text{ and } r_2^2 \cos^2\theta = 4ar_2 \sin\theta$$

$$\Rightarrow \quad r_1 = \frac{4a\cos\theta}{\sin^2\theta} \text{ and } r_2 = \frac{4a\sin\theta}{\cos^2\theta}$$

$$\therefore \quad (r_1 r_2)^{4/3} = \left(\frac{4a\cos\theta}{\sin^2\theta}.\frac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3} \qquad \text{....... (i)}$$

$$\text{and} \quad 16a^2.\left(r_1^{2/3} + r_2^{2/3}\right) = 16a^2\left\{\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^{2/3} + \left(\frac{4a\sin\theta}{\cos^2\theta}\right)^{2/3}\right\}$$

$$= 16a^2.(4a)^{2/3}\left\{\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{4/3}} + \frac{(\sin\theta)^{2/3}}{(\cos\theta)^{4/3}}\right\} = 16a^2.(4a)^{2/3}\left\{\frac{\cos^2\theta + \sin^2\theta}{(\sin\theta)^{4/3}(\cos\theta)^{4/3}}\right\}$$

$$= \frac{16a^2.(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}} = \left(\frac{16a^2}{\cos\theta\cos\theta}\right)^{4/3} = (r_1 r_2)^{4/3} \qquad \{\text{from (i)}\}$$

Illustration 26 : The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Solution : Let the three points on the parabola be

$(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$

The area of the triangle formed by these points

$$\Delta_1 = \frac{1}{2}[at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_2^2(2a_1 - 2at_2)]$$

$= -a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)$.

The points of intersection of the tangents at these points are

$(at_2t_3,\ a(t_2 + t_3))$, $(at_3t_1,\ a(t_3 + t_1))$ and $(at_1t_2,\ a(t_1 + t_2))$

The area of the triangle formed by these three points

$$\Delta_2 = \frac{1}{2}\left\{at_2t_3(at_3 - at_2) + at_3t_1(at_1 - at_3) + at_1t_2(at_2 - at_1)\right\}$$

$$= \frac{1}{2}a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)$$

Hence $\Delta_1 = 2\Delta_2$

Illustration 27 : Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

Solution : Let the equations of the three tangents be

$$t_1 y = x + at_1^2 \qquad \text{................(i)}$$

$$t_2 y = x + at_2^2 \qquad \text{................(ii)}$$

$$\text{and} \quad t_3 y = x + at_3^2 \qquad \text{................(iii)}$$

The point of intersection of (ii) and (iii) is found, by solving them, to be $(at_2t_3,\ a(t_2 + t_3))$

The equation of the straight line through this point & perpendicular to (i) is

$$y - a(t_2 + t_3) = -t_1(x - at_2t_3)$$

i.e. $y + t_1x = a(t_2 + t_3 + t_1t_2t_3)$ ....(iv)

Similarly, the equation of the straight line through the point of intersection of (iii) and (i) & perpendicular to (ii) is

$y + t_2x = a(t_3 + t_1 + t_1t_2t_3)$ ....(v)

and the equation of the straight line through the point of intersection of (i) and (ii) & perpendicular to (iii) is

$y + t_1x = a(t_1 + t_2 + t_1t_2t_3)$ ....(vi)

The point which is common to the straight lines (iv), (v) and (vi)

i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$$x = -a,\ y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$$

and this point lies on the directrix.

ANSWERS FOR DO YOURSELF

1 : (i) Parabola (ii) Vertex : $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Axis : $y = \frac{5}{2}$, Focus : $\left(-\frac{17}{4}, \frac{5}{2}\right)$, Directrix : $x = -\frac{11}{4}$; LR = 3

(iii) $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$; Axis : $2x - y = 3$; LR $= 4\sqrt{5}$ unit

(iv) $(3x + 4y - 4)^2 = 20(4x - 3y + 7)$

2 : (i) $\left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$ (ii) $(-7, 8), (9, 8)$ (iv) $kb^2 = ac$ (v) $8\sqrt{3}$

3 : (i) $x - y + 3 = 0, (3, 6)$; $3x - 2y + 4 = 0, \left(\frac{4}{3}, 4\right)$

(ii) $2x - y + 2 = 0$, $(1, 4)$; $x + 2y + 16 = 0$, $(16, -16)$

4 : (i) C (ii) 1 (iii) 48 (iv) $(0, 0), (-4, 3)$ and $(16, 8)$

5 : (i) $(y + 2)^2 = 28(x - 1)$ (ii) $\pi/2$

6 : (i) $2x = y + 1$ (ii) $(14, 12)$ (iii) $y^2 = 2a(x - a)$

7 : (i) 1 : 1 : 1 (ii) $y^2 - 4x + 2 = 0$

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. Latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7 + 5\sqrt{2}$ is -

(A) 5 (B) 10 (C) 20 (D) 15

2. Directrix of a parabola is x + y = 2. If it's focus is origin, then latus rectum of the parabola is equal to -

(A) $\sqrt{2}$ units (B) 2 units (C) $2\sqrt{2}$ units (D) 4 units

3. Which one of the following equations represents parametrically, parabolic profile ?

(A) $x = 3\cos t$; $y = 4\sin t$ (B) $x^2 - 2 = -\cos t$; $y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$ (D) $x = \sqrt{1 - \sin t}$; $y = \sin\frac{t}{2} + \cos\frac{t}{2}$

4. Let C be a circle and L a line on the same plane such that C and L do not intersect. Let P be a moving point such that the circle drawn with centre at P to touch L also touches C. Then the locus of P is -

(A) a straight line parallel to L not intersecting C

(B) a circle concentric with C

(C) a parabola whose focus is centre of C and whose directrix is L.

(D) a parabola whose focus is the centre of C and whose directrix is a straight line parallel to L.

5. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be-

(A) $\left(t + \frac{1}{t}\right)^2$ (B) $\left(t + \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$ (C) $\left(t - \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$ (D) none

6. From the focus of the parabola $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex and focus of the parabola. The equation of the circle is -

(A) $(x - 2)^2 + y^2 = 3$ (B) $(x - 2)^2 + y^2 = 9$ (C) $(x + 2)^2 + y^2 = 9$ (D) $x^2 + y^2 - 4x = 0$

7. The point of intersection of the curves whose parametric equations are $x = t^2+1$, $y = 2t$ and $x = 2s$, $y = 2/s$ is given by -

(A) (4, 1) (B) (2, 2) (C) (–2, 4) (D) (1, 2)

8. If M is the foot of the perpendicular from a point P of a parabola $y^2= 4ax$ to its directrix and SPM is an equilateral triangle, where S is the focus, then SP is equal to -

(A) a (B) 2a (C) 3a (D) 4a

9. Through the vertex 'O' of the parabola $y^2 = 4ax$, variable chords OP and OQ are drawn at right angles. If the variable chord PQ intersects the axis of x at R, then distance OR :

(A) varies with different positions of P and Q (B) equals the semi latus rectum of the parabola

(C) equals latus rectum of the parabola (D) equals double the latus rectum of the parabola

10. The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the points Q and R is -

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$ (C) $\frac{2A}{a}$ (D) $\frac{4A}{a}$

11. Point P lies on $y^2 = 4ax$ & N is foot of perpendicular from P on its axis. A straight line is drawn parallel to the axis to bisect NP and meets the curve in Q. NQ meets the tangent at the vertex in a point T such that AT = k NP, then the value of k is : (where A is the vertex)

(A) 3/2 (B) 2/3 (C) 1 (D) none

12. The tangents to the parabola $x = y^2 + c$ from origin are perpendicular then c is equal to -

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{4}$

13. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is -

(A) $y^2 = \frac{9}{2}ax$ (B) $y^2 = \frac{9}{4}ax$ (C) $y^2 = 9ax$ (D) $x^2 = 4ay$

14. T is a point on the tangent to a parabola $y^2 = 4ax$ at its point P. TL and TN are the perpendiculars on the focal radius SP and the directrix of the parabola respectively. Then -

(A) SL = 2 (TN) (B) 3 (SL) = 2 (TN) (C) SL = TN (D) 2 (SL) = 3 (TN)

15. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

(A) $x^2 + y^2 - 4y = 0$ (B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$ (D) $x^2 + y^2 - 2x - 4y + 1 = 0$

16. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is-

(A) 8 (B) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

17. Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. The length , these tangents will intercept on the line x = 2 :

(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these

18. Locus of the point of intersection of the perpendiculars tangent of the curve $y^2 + 4y - 6x - 2 = 0$ is :

(A) $2x - 1 = 0$ (B) $2x + 3 = 0$ (C) $2y + 3 = 0$ (D) $2x + 5 = 0$

19. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are-

(A) (3, 2) (B) (2, 4) (C) (3, 4) (D) (4, 1)

20. The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A & B. The co-ordinates of the point of intersection of the tangents drawn at the points A & B are :

(A) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (B) $\left(-\frac{5}{2}, \frac{7}{2}\right)$ (C) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (D) $\left(-\frac{7}{2}, \frac{5}{2}\right)$

21. From the point (4, 6) a pair of tangent lines are drawn to the parabola, $y^2 = 8x$. The area of the triangle formed by these pair of tangent lines & the chord of contact of the point (4, 6) is

(A) 2 (B) 4 (C) 8 (D) none

22. TP & TQ are tangents to the parabola, $y^2=4ax$ at P & Q. If the chord PQ passes through the fixed point (-a, b) then the locus of T is -

(A) $ay = 2b (x - b)$ (B) $bx = 2a (y - a)$ (C) $by = 2a (x - a)$ (D) $ax = 2b (y - b)$

23. If the tangent at the point P (x_1,y_1) to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q & R, then the mid point of QR is -

(A) $(x_1+ b, y_1+ b)$ (B) $(x_1- b, y_1- b)$ (C) (x_1, y_1) (D) $(x_1+ b, y_1)$

24. Let PSQ be the focal chord of the parabola, $y^2 = 8x$. If the length of SP=6 then, l(SQ) is equal to(where S is the focus) -

(A) 3 (B) 4 (C) 6 (D) none

25. Two parabolas $y^2 = 4a(x - l_1)$ and $x^2 = 4a(y - l_2)$ always touch one another , the quantities l_1 and l_2 are both variable. Locus of their point of contact has the equation -

(A) $xy = a^2$ (B) $xy = 2a^2$ (C) $xy = 4a^2$ (D) none

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

26. Equation $x^2 - 2x - 2y + 5 = 0$ represents -

(A) a parabola with vertex (1, 2) (B) a parabola with vertex (2, 1)

(C) a parabola with directrix $y = \frac{3}{2}$ (D) a parabola with directrix $y = \frac{2}{5}$

27. The normals to the parabola $y^2 = 4ax$ from the point (5a, 2a) are -

(A) $y = -3x + 33a$ (B) $x = -3y + 3a$ (C) $y = x - 3a$ (D) $y = -2x + 12a$

28. The equation of the lines joining the vertex of the parabola $y^2 = 6x$ to the points on it whose abscissa is 24, is -

(A) $2y + x + 1 = 0$ (B) $2y - x + 1 = 0$ (C) $x + 2y = 0$ (D) $x - 2y = 0$

29. The equation of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) is -

(A) $x + 4y + 1 = 0$ (B) $x - 4y + 36 = 0$ (C) $9x - 4y + 4 = 0$ (D) $9x + 4y + 4 = 0$

30. Consider the equation of a parabola $y^2 = 4ax$, $(a < 0)$ which of the following is false -

(A) tangent at the vertex is $x = 0$ (B) directrix of the parabola is $x = 0$

(C) vertex of the parabola is at the origin (D) focus of the parabola is at $(-a, 0)$

CHECK YOUR GRASP — **ANSWER KEY** — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	C	B	D	A	B	B	D	C	C	B	D	A	C	D
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	B	B	D	C	C	A	C	C	A	C	A,C	C,D	C,D	B,C	B,D

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is -

(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$ (C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

2. Let A be the vertex and L the length of the latus rectum of parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with point A as vertex, 2L as the length of the latus rectum and the axis at right angles to that of the given curve is -

(A) $x^2 + 4x + 8y - 4 = 0$ (B) $x^2 + 4x - 8y + 12 = 0$

(C) $x^2 + 4x + 8y + 12 = 0$ (D) $x^2 + 8x - 4y + 8 = 0$

3. The parametric coordinates of any point on the parabola $y^2 = 4ax$ can be -

(A) $(at^2, 2at)$ (B) $(at^2, -2at)$ (C) $(a\sin^2 t, 2a\sin t)$ (D) $(a\sin t, 2a\cos t)$

4. PQ is a normal chord of the parabola $y^2 = 4ax$ at P, A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x-axis in R. Then the length of of AR is -

(A) equal to the length of the latus rectum

(B) equal to the focal distance of the point P.

(C) equal to twice the focal distance of the point P.

(D) equal to the distance of the point P from the directrix

5. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is-

(A) $5\sqrt{2}$ (B) $4\sqrt{5}$ (C) $4\sqrt{50}$ (D) $2\sqrt{5}$

6. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then -

(A) $x_1 = x_2$ (B) $x_1 = y_2$ (C) $y_1 = y_2$ (D) $x_2 = y_1$

7. Locus of the intersection of the tangents at the ends of the normal chords of the parabola $y^2 = 4ax$ is -

(A) $(2a + x)y^2 + 4a^3 = 0$ (B) $(x + 2a)y^2 + 4a^2 = 0$

(C) $(y + 2a)x^2 + 4a^3 = 0$ (D) none

8. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(A) latus rectum is half the latus rectum of the original parabola

(B) vertex is (a/2, 0)

(C) directrix is y-axis

(D) focus has the co-ordinates (a, 0)

9. The equation of a straight line passing through the point (3, 6) and cutting the curve $y = \sqrt{x}$ orthogonally is -

(A) $4x + y - 18 = 0$ (B) $x + y - 9 = 0$ (C) $4x - y - 6 = 0$ (D) none

10. The tangent and normal at P (t), for all real positive t, to the parabola $y^2 = 4ax$ meet the axis of the parabola in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through the points P, T and G is -

(A) $\cot^{-1}t$ (B) $\cot^{-1}t^2$ (C) $\tan^{-1}t$ (D) $\sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

11. A variable circle is described to passes through the point (1, 0) and tangent to the curve $y = \tan(\tan^{-1}x)$. The locus of the centre of the circle is a parabola whose -

(A) length of the latus rectum is $2\sqrt{2}$ (B) axis of symmetry has the equation $x + y = 1$

(C) vertex has the co-ordinates (3/4, 1/4) (D) none of these

12. AB, AC are tangents to a parabola $y^2 = 4ax$. p_1 p_2 and p_3 are the lengths of the perpendiculars from A, B and C respectively on any tangent to the curve, then p_2, p_1, p_3 are in-

(A) A.P. (B) G.P. (C) H.P. (D) none of these

13. Through the vertex O of the parabola, $y^2 = 4ax$ two chords OP and OQ are drawn and the circles on OP and OQ as diameter intersect in R. If θ_1, θ_2 and ϕ are the angles made with the axis by the tangent at P and Q on the parabola and by OR then the value of $\cot\theta_1 + \cot\theta_2 =$

(A) $-2\tan\phi$ (B) $-2\tan(\pi - \phi)$ (C) 0 (D) $2\cot\phi$

14. Two parabolas have the same focus. If their directrices are the x-axis & the y-axis respectively, then the slope of their common chord is -

(A) 1 (B) –1 (C) 4/3 (D) 3/4

15. Tangent to the parabola $y^2 = 4ax$ at point P meets the tangent at vertex A, at point B and the axis of parabola at T. Q is any point on this tangent and N is the foot of perpendicular from Q on SP, where S is focus. M is the foot of perpendicular from Q on the directrix then -

(A) B bisects PT (B) B trisects PT (C) QM = SN (D) QM = 2SN

16. If the distance between a tangent to the parabola $y^2 = 4x$ and a parallel normal to the same parabola is $2\sqrt{2}$, then possible values of gradient of either of them are -

(A) –1 (B) +1 (C) $-\sqrt{\sqrt{5}-2}$ (D) $+\sqrt{\sqrt{5}-2}$

17. If two distinct chords of a parabola $x^2 = 4ay$ passing through (2a, a) are bisected on the line $x + y = 1$, then length of latus rectum can be -

(A) 2 (B) 1 (C) 4 (D) 5

18. If PQ is a chord of parabola $x^2 = 4y$ which subtends right angle at vertex. Then locus of centroid of triangle PSQ (S is focus) is a parabola whose -

(A) vertex is (0, 3) (B) length of LR is 4/3

(C) axis is x = 0 (D) tangent at the vertex is x = 3

19. Identify the correct statement(s) -

(A) In a parabola vertex is the mid point of focus and foot of directrix.

(B) $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$ are two points on $y^2 = 4ax$ such that $t_1t_2 = -1$, then normals at P and Q are perpendicular.

(C) There doesn't exist any tangent of $y^2 = 4ax$ which is parallel to x-axis.

(D) At most two normals can be drawn to a parabola from any point on its plane.

20. For parabola $y^2 = 4ax$ consider three points A, B, C lying on it. If the centroid of ΔABC is (h_1, k_1) & centroid of triangle formed by the point of intersection of tangents at A, B, C has coordinates (h_2, k_2), then which of the following is always true -

(A) $2k_1 = k_2$ (B) $k_1 = k_2$ (C) $k_1^2 = \frac{4a}{3}(h_1 + 2h_2)$ (D) $k_1^2 = \frac{4a}{3}(2h_1 + h_2)$

BRAIN TEASERS — **ANSWER KEY** — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A,B	A,B	C	D	C	A	A,B,C,D	A	C,D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B,C	B	A	A,B	A,C	A,B,C,D	A,B	A,B,C	A,B,C	B,C

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-II
(A)	The normal chord at a point t on the parabola $y^2 = 4x$ subtends a right angle at the vertex, then t^2 is	(p)	4
(B)	The area of the triangle inscribed in the curve $y^2 = 4x$. If the parameter of vertices are 1, 2 and 4 is	(q)	2
(C)	The number of distinct normal possible from $\left(\frac{11}{4}, \frac{1}{4}\right)$ to the parabola $y^2 = 4x$ is	(r)	3
(D)	The normal at (a, 2a) on $y^2 = 4ax$ meets the curve again at $(at^2, 2at)$, then the value of $\lvert t - 1\rvert$ is	(s)	6

2.

	Column-I		Column-II
(A)	Area of a triangle formed by the tangents drawn from a point (–2, 2) to the parabola $y^2 = 4(x + y)$ and their corresponding chord of contact is	(p)	8
(B)	Length of the latus rectum of the conic $25\{(x - 2)^2 + (y - 3)^2\} = (3x + 4y - 6)^2$ is	(q)	$4\sqrt{3}$
(C)	If focal distance of a point on the parabola $y = x^2 - 4$ is 25/4 and points are of the form $(\pm\sqrt{a}, b)$ then value of a + b is	(r)	4
(D)	Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$ whose one angular point is vertex of the parabola, is	(s)	24/5

ASSERTION & REASON

These questions contain, Statement-I (assertion) and Statement-II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for Statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : If normal at the ends of double ordinate x = 4 of parabola $y^2 = 4x$ meet the curve again at P and P' respectively, then PP' = 12 unit.

Because

Statement-II : If normal at t_1 of $y^2 = 4ax$ meets the parabola again at t_2, then $t_1^2 = 2 + t_1 t_2$.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The lines from the vertex to the two extremities of a focal chord of the parabola $y^2 = 4ax$ are at an angle of $\frac{\pi}{2}$.

Because

Statement-II : If extremities of focal chord of parabola are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, then $t_1t_2 = -1$

(A) A (B) B (C) C (D) D

3. **Statement-I** : If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$, then the locus of point of intersection of chords P_1P_2 and Q_1Q_2 is directrix of the parabola. Here P_1P_2 and Q_1Q_2 are not parallel.
Because
Statement-II : The locus of point of intersection of perpendicular tangents of parabola is directrix of parabola.
(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Observe the following facts for a parabola :

(i) Axis of the parabola is the only line which can be the perpendicular bisector of the two chords of the parabola.

(ii) If AB and CD are two parallel chords of the parabola and the normals at A and B intersect at P and the normals at C and D intersect at Q, then PQ is a normal to the parabola.

Let a parabola is passing through (0, 1), (–1, 3), (3, 3) & (2, 1)

On the basis of above information, answer the following questions :

1. The vertex of the parabola is -
(A) $\left(1, \frac{1}{3}\right)$ (B) $\left(\frac{1}{3}, 1\right)$ (C) (1, 3) (D) (3, 1)

2. The directrix of the parabola is -
(A) $y - \frac{1}{24} = 0$ (B) $y + \frac{1}{2} = 0$ (C) $y + \frac{1}{24} = 0$ (D) $y + \frac{1}{12} = 0$

3. For the parabola $y^2 = 4x$, AB and CD are any two parallel chords having slope 1. C_1 is a circle passing through O, A and B and C_2 is a circle passing through O, C and D, where O is origin. C_1 and C_2 intersect at -
(A) (4, –4) (B) (–4, 4) (C) (4, 4) (D) (–4, –4)

Comprehension # 2 :

If a source of light is placed at the fixed point of a parabola and if the parabola is a reflecting surface, then the ray will bounce back in a line parallel to the axis of the parabola.

On the basis of above information, answer the following questions :

1. A ray of light is coming along the line y = 2 from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 8x$, then the equation of the reflected ray is -
(A) $2x + 5y = 4$ (B) $3x + 2y = 6$ (C) $4x + 3y = 8$ (D) $5x + 4y = 10$

2. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $y^2 + 10y - 4x + 17 = 0$ After reflection, the ray must pass through the point -
(A) (–2, –5) (B) (–1, –5) (C) (–3, –5) (D) (–4, –5)

3. Two ray of light coming along the lines y = 1 and y = –2 from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = x$ at A and B respectively. The reflected rays pass through a fixed point C, then the area of the triangle ABC is -
(A) $\frac{21}{8}$ sq. unit (B) $\frac{19}{2}$ sq. unit (C) $\frac{17}{2}$ sq. unit (D) $\frac{15}{2}$ sq. unit

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE -3

- ***Match the Column***

1. (A) → (q); (B) → (s); (C) → (q); (D) → (p) 2. (A) → (r); (B) → (s); (C) → (p); (D) → (q)

- ***Assertion & Reason***

1. C 2. D 3. B

- ***Comprehension Based Questions***

Comprehension # 1 : 1. A 2. C 3. A

Comprehension # 2 : 1. C 2. B 3. A

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the equation of parabola, whose focus is $(-3, 0)$ and directrix is $x + 5 = 0$.

2. Find the vertex, axis, focus, directrix, latus rectum of the parabola $x^2 + 2y - 3x + 5 = 0$

3. Find the equation of the parabola whose focus is $(1, -1)$ and whose vertex is $(2, 1)$. Also find its axis and latus rectum.

4. If the end points $P(t_1)$ and $Q(t_2)$ of a chord of a parabola $y^2 = 4ax$ satisfy the relation $t_1t_2 = k$ (constant) then prove that the chord always passes through a fixed point. Find that point also ?

5. Find the locus of the middle points of all chords of the parabola $y^2 = 4ax$ which are drawn through the vertex.

6. O is the vertex of the parabola $y = 4ax$ & L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H , prove that the length of the double ordinate through H is $4a\sqrt{5}$.

7. Find the length of the side of an equilateral triangle inscribed in the parabola, $y^2 = 4x$ so that one of its angular point is at the vertex.

8. Two perpendicular chords are drawn from the origin 'O' to the parabola $y = x^2$, which meet the parabola at P and Q. Rectangle POQR is completed. Find the locus of vertex R.

9. Find the set of values of α in the interval $[\pi/2, 3\pi/2]$, for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$.

10. Find the length of the focal chord of the parabola $y^2 = 4ax$ whose distance from the vertex is p.

11. If 'm' varies then find the range of c for which the line $y = mx + c$ touches the parabola $y^2 = 8(x + 2)$.

12. Find the equations of the tangents to the parabola $y = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact .

13. Find the equations of the tangents of the parabola $y = 12x$, which passes through the point $(2, 5)$.

14. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y(2x + a) = a(3x + a)$.

15. Two tangents to the parabola $y = 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y = 8(x + 2)$.

16. Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point $(6, 9)$.

17. In the parabola $y = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T & the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5.

18. Show that the normals at the points $(4a, 4a)$ & at the upper end of the latus rectum of the parabola $y = 4ax$ intersect on the same parabola.

19. Show that the locus of a point, such that two of the three normals drawn from it to the parabola $y = 4ax$ are perpendicular is $y = a(x - 3a)$.

20. If the normal at $P(18, 12)$ to the parabola $y = 8x$ cuts it again at Q, then show that $9PQ = 80\sqrt{10}$

21. Prove that the locus of the middle point of portion of a normal to $y = 4ax$ intercepted between the curve & the axis is another parabola. Find the vertex & the latus rectum of the second parabola.

22. A variable chord PQ of the parabola $y = 4x$ is drawn parallel to the line $y = x$. If the parameters of the points P & Q on the parabola are p & q respectively, show that $p + q = 2$. Also show that the locus of the point of intersection of the normals at P & Q is $2x - y = 12$.

23. P & Q are the points of contact of the tangents drawn from the point T to the parabola $y = 4ax$. If PQ be the normal to the parabola at P, prove that TP is bisected by the directrix.

24. The normal at a point P to the parabola $y = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that QG − PG = constant.

25. Three normals to $y = 4x$ pass through the point (15, 12). Show that one of the normals is given by $y = x - 3$ & find the equations of the others.

CONCEPTUAL SUBJECTIVE EXERCISE — **ANSWER KEY** — EXERCISE-4(A)

1. $y^2 = 4(x + 4)$

2. Vertex $\equiv \left(\frac{3}{2}, \frac{-11}{8}\right)$, focus $\equiv \left(\frac{3}{2}, \frac{-15}{8}\right)$, axis : $x = \frac{3}{2}$, directrix : $y = -\frac{7}{8}$, latus rectum = 2

3. $(2x - y - 3)^2 = -20(x + 2y - 4)$, axis : $2x - y - 3 = 0$. latus rectum $= 4\sqrt{5}$.

4. $(-ak, 0)$ **5.** $y^2 = 2ax$ **7.** $8\sqrt{3}$ **8.** $x^2 = y - 2$

9. $\alpha \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$ **10.** $\frac{4a^3}{p^2}$ **11.** $(-\infty, -4] \cup [4, \infty)$

12. $2x - y + 2 = 0$, (1, 4) ; $x + 2y + 16 = 0$, (16, −16) **13.** $3x - 2y + 4 = 0$; $x - y + 3 = 0$

16. $x^2 + y^2 + 18x - 28y + 27 = 0$ **21.** (a , 0) ; a **25.** $y = -4x + 72$, $y = 3x - 33$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. If from the vertex of a parabola a pair of chords be drawn at right angles to one another, & with these chords as adjacent sides a rectangle be constructed , then find the locus of the outer corner of the rectangle.

2. Two perpendicular straight lines through the focus of the parabola $y = 4ax$ meet its directrix in T & T′ respectively. Show that the tangents to the parabola parallel to the perpendicular lines intersect in the mid point of T T'.

3. Find the condition on 'a' & 'b' so that the two tangents drawn to the parabola $y = 4ax$ from a point are normals to the parabola $x = 4by$.

4. TP & TQ are tangents to the parabola and the normals at P & Q meet at a point R on the curve. Prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y = a(x - a)$.

5. Let S is the focus of the parabola $y^2 = 4ax$ and X the foot of the directrix, PP' is a double ordinate of the curve and PX meets the curve again in Q. Prove that P'Q passes through focus.

6. Prove that on the axis of any parabola $y = 4ax$ there is a certain point K which has the property that , if a chord PQ of the parabola be drawn through it , then $\frac{1}{(PK)^2} + \frac{1}{(QK)^2}$ is same for all positions of the chord. Find also the coordinates of the point K.

7. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points on the parabola $y^2 = 4ax$ and the normals at these points meet in a point, then prove that $\frac{x_1 - x_2}{y_3} + \frac{x_2 - x_3}{y_1} + \frac{x_3 - x_1}{y_2} = 0$

8. A variable chord joining points $P(t_1)$ and $Q(t_2)$ of the parabola $y = 4ax$ subtends a right angle at a fixed point t_0 of the curve. Show that it passes through a fixed point. Also find the co-ordinates of the fixed point.

9. Show that a circle circumscribing the triangle formed by three co-normal points passes through the vertex of the parabola and its equation is, $2(x^2+y^2) - 2(h+2a)x - ky = 0$, where (h, k) is the point from where three concurrent normals are drawn.

10. A ray of light is coming along the line y = b from the positive direction of x-axis & strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray & show that it passes through the focus of the parabola. Both a & b are positive. **[REE 95]**

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)

1. $y = 4a(x - 8a)$ **3.** $a > 8b$ **6.** $(2a, 0)$ **8.** $[a(t_0 + 4), -2at_0]$ **10.** $4abx + (4a - b)y - 4ab = 0$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The length of the latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is- [AIEEE-2002]
(1) 4 (2) 6 (3) 8 (4) 10

2. The equation of tangents to the parabola $y^2 = 4ax$ at the ends of its latus rectum is- [AIEEE-2002]
(1) $x - y + a = 0$ (2) $x + y + a = 0$ (3) $x + y - a = 0$ (4) both (1) and (2)

3. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then- [AIEEE-2003]
(1) $t_2 = t_1 + \frac{2}{t_1}$ (2) $t_2 = -t_1 - \frac{2}{t_1}$ (3) $t_2 = -t_1 + \frac{2}{t_1}$ (4) $t_2 = t_1 - \frac{2}{t_1}$

4. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then- [AIEEE-2004]
(1) $d^2 + (2b + 3c)^2 = 0$ (2) $d^2 + (3b + 2c)^2 = 0$ (3) $d^2 + (2b - 3c)^2 = 0$ (4) $d^2 + (3b - 2c)^2 = 0$

5. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is- [AIEEE-2006]
(1) $xy = \frac{3}{4}$ (2) $xy = \frac{35}{16}$ (3) $xy = \frac{64}{105}$ (4) $xy = \frac{105}{64}$

6. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangents is- [AIEEE-2007]
(1) $(-1, 1)$ (2) $(0, 2)$ (3) $(2, 4)$ (4) $(-2, 0)$

7. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at - [AIEEE-2008]
(1) $(0, 2)$ (2) $(1, 0)$ (3) $(0, 1)$ (4) $(2, 0)$

8. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is :- [AIEEE-2010]
(1) $x = 1$ (2) $2x + 1 = 0$ (3) $x = -1$ (4) $2x - 1 = 0$

9. Given : A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}\,x$. [JEE (Main)-2013]

Statement–I : An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement–II : If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true, Statement-II is false.
(4) Statement-I is false, Statement-II is true.

PREVIOUS YEARS QUESTIONS — ANSWER KEY — EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9
Ans	3	4	2	1	4	4	2	3	2

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. **(a)** If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of 'k' is:

(A) 1/8 (B) 8 (C) 4 (D) 1/4

(b) If $x + y = k$ is normal to $y^2 = 12x$, then 'k' is - **[JEE 2000 (Screening) 1+1M]**

(A) 3 (B) 9 (C) – 9 (D) – 3

2. **(a)** The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x- axis is -

(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x + 3)$ (C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x + 1)$

(b) The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is - **[JEE 2001 (Screening) 1+1M]**

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

3. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix **[JEE 2002 (Screening), 3M]**

(A) $x = -a$ (B) $x = -\frac{a}{2}$ (C) $x = 0$ (D) $x = \frac{a}{2}$

4. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is - **[JEE 2002 (Scr), 3M]**

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$

5. If a focal chord of the parabola $y^2 = 16x$ is a tangent to the circle $(x - 6)^2 + y^2 = 2$. then the set of possible values of the slope of this chord, are - **[JEE 2003 (Scr), 3M]**

(A) $\{-1, 1\}$ (B) $\{-2, 2\}$ (C) $\left\{-2, \frac{1}{2}\right\}$ (D) $\left\{2, -\frac{1}{2}\right\}$

6. Normals with slopes m_1, m_2, m_3 are drawn from the point P to the parabola $y^2 = 4x$. If locus of P with $m_1 m_2 = \alpha$ is a part of the parabola itself, find α. **[JEE 2004 (Mains), 4M out of 60]**

7. Two tangents are drawn from point (1, 4) to the parabola $y^2 = 4x$. Angles between tangents is -

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

[JEE 2004 (Screening), 3M]

8. At any point P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q. Find the locus of point R which divides QP externally in the ratio $\frac{1}{2} : 1$. **[JEE 2004 (Mains), 4M out of 60]**

9. Tangent to the curve $y = x^2 + 6$ at point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then coordinate of Q is - **[JEE 2005 (Screening) 3M]**

(A) (–6, 11) (B) (6, –11) (C) (–6, –7) (D) (–6, –11)

10. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that of origin from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, then the equation of the parabola is - **[JEE 2006 (3M, –1M) out of 184]**

(A) $(x + y)^2 = (x - y - 2)$ (B) $(x - y)^2 = (x + y - 2)$

(C) $(x - y)^2 = 4(x + y - 2)$ (D) $(x - y)^2 = 8(x + y - 2)$

11. The equations of the common tangents to the parabola $y = x^2$ and $y = -x^2 + 4x - 4$ is/are-

(A) $y = 4(x - 1)$ (B) $y = 0$ (C) $y = -4(x - 1)$ (D) $y = -30x - 50$

[JEE 2006 , (5M, –1M) out of 184]

12. **Match the following** **[JEE 2006, (6M, 0M) out of 184]**

Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then

(i)	Area of ΔPQR	(A)	2
(ii)	Radius of circumcircle of ΔPQR	(B)	5/2
(iii)	Centroid of ΔPQR	(C)	(5/2, 0)
(iv)	Circumcentre of ΔPQR	(D)	(2/3, 0)

13 to 15 are based on this paragraph **[JEE 2006 (5M, –2M) each, out of 184]**

Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A.

13. If P is a point on C_1 and Q in another point on C_2, then $\dfrac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to -

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

14. A circle touches the line L and circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is -

(A) ellipse (B) hyperbola (C) parabola (D) pair of straight line

15. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1 then area of $\Delta T_1T_2T_3$ is

(A) 1/2 sq. units (B) 2/3 sq. units (C) 1 sq. units (D) 2 sq. units

16 to 18 are based on this paragraph

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

16. The ratio of the areas of the triangle PQS and PQR is :- **[JEE 2007, 4M]**

(A) $1 : \sqrt{2}$ (B) 1 : 2 (C) 1 : 4 (D) 1 : 8

17. The radius of the circumcircle of the triangle PRS is :- **[JEE 2007, 4M]**

(A) 2 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

18. The radius of the incircle of the triangle PQR is :- **[JEE 2007, 4M]**

(A) 4 (B) 3 (C) $\dfrac{8}{3}$ (D) 2

Assertion and Reason :

19. **Statement-1** : The curve $y = \dfrac{-x^2}{2} + x + 1$ is symmetric with respect to the line x = 1 because

Statement-2 : A parabola is symmetric about its axis. **[JEE 2007, 3M]**

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

20. Consider the two curves $C_1 : y^2 = 4x$; $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then **[JEE 2008, 3M, –1M]**

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

21. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose **[JEE 2009, 4M, –1M]**

(A) vertex is $\left(\dfrac{2a}{3},0\right)$ (B) directrix is x =0 (C) latus rectum is $\dfrac{2a}{3}$ (D) focus is (a, 0)

22. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be - **[JEE 2010, 3M]**

(A) –1/r (B) 1/r (C) 2/r (D) –2/r

23. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\dfrac{1}{2},2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\dfrac{\Delta_1}{\Delta_2}$ is **[JEE 2011,4M]**

24. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0,0) to (x,y) in the ratio 1 : 3. Then the locus of P is - **[JEE 2011,3M]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

25. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by - **[JEE 2011,4M]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

26. Let S be the focus of the parabola $y^2 = 8x$ & let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[JEE 2012, 4M]**

Paragraph for Question 27 and 28

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

27. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then tanθ = **[JEE(Advanced) 2013, 3, (–1)M]**

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

28. Length of chord PQ is **[JEE(Advanced) 2013, 3, (–1)M]**

(A) 7a (B) 5a (C) 2a (D) 3a

29. A line L : $y = mx + 3$ meets y-axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0,y_0)$. The tangent to the parabola at $F(x_0,y_0)$ intersects the y-axis at $G(0,y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		List-II
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

[JEE(Advanced) 2013, 3, (–1)M]

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. (a) C (b) B	2. (a) C (b) D	3. C	4. D	5. A	6. 2	7. C	8. $(x + 1)(y - 1)^2 + 4 = 0$
9. C	10. D	11. A, B	12. (i) A, (ii) B, (iii) D, (iv) C	13. A	14. C	15. C	
16. C	17. B	18. D	19. A	20. B	21. A,D	22. C, D	23. 2
24. C	25. A,B,D	26. 4	27. D	28. B	29. A		